A unified approach to Aharonov-Bohm, Aharonov-Casher and which-path experiments

1999 J. Phys. A: Math. Gen. 32 5517

(http://iopscience.iop.org/0305-4470/32/29/312)

J. Phys. A: Math. Gen. **32** (1999) 5517–5528. Printed in the UK PII: S0305-4470(99)02405-1

A unified approach to Aharonov–Bohm, Aharonov–Casher and which-path experiments

A Vourdas

Department of Electrical Engineering and Electronics, University of Liverpool, Brownlow Hill, Liverpool L69 3GJ, UK

Received 8 March 1999, in final form 5 May 1999

Abstract. A unified approach to Aharonov–Bohm, Aharonov–Casher and which-path experiments is presented, using an enlarged Hilbert space. This Hilbert space contains quasi-periodic Aharonov–Bohm wavefunctions $R(x + 2\pi) = R(x)\exp(\mathrm{i}\theta)$ with various values of θ. Thus it can describe which-path Aharonov–Bohm experiments where the phase θ is uncertain due to decoherence that occurs as a result of the observation of the paths of the electric charges. The same Hilbert space contains quasi-periodic Aharonov–Casher wavefunctions which describe magnetic flux tubes winding around an electric charge and which are related through a Fourier transform to the Aharonov–Bohm wavefunctions. The duality between these two phenomena is discussed. The decoherence occuring in which-path experiments is studied quantitatively. Magnetic and electric superselection rules, appropriate for the Aharonov–Bohm and Aharonov–Casher experiments correspondingly, are also discussed.

1. Introduction

The Aharonov–Bohm effect [1–4], studies the interference of electrons that follow two different paths in the presence of a magnetostatic flux ϕ which threads the area between the two paths. This work provides the theoretical background for the understanding of many other phenomena (e.g. magnetoresistance oscillations in small rings [5]).

When such an experiment is performed there are two extreme cases. In one the phase $\theta = e\phi$ has a certain value; while the dual variable which as we will explain is the charge (or winding number) w, has large uncertainty. This is realized in Aharonov–Bohm type of experiments, where the paths of the particles are not observed and we get interference.

In the second extreme case, we have large uncertainty in θ and a certain value of w. The experiment now contains a mechanism which observes the paths of the particles. The phase θ is disturbed by the measuring apparatus, decoherence occurs and the interference is destroyed. This has been discussed by several authors [6–8]. More recently it has also been discussed in the context of 'which path' experiments in [9]. Similar experiments have been recently performed using a mesoscopic ring with a quantum dot, which interacts with a quantum point contact [10, 11]. The latter plays the role of an observer which determines the path of the electrons and thus destroys their interference.

Apart from the above two extreme cases, there are in between situations where we have some uncertainty in the phase θ (partial decoherence), some uncertainty in the charge (winding number) w, and partial destruction of the interference.

The Aharonov–Casher effect has also been studied extensively in the literature both theoretically [12] and experimentally [13, 14]. In contrast to the Aharonov–Bohm effect where

0305-4470/99/295517+12$30.00

electric charges wind around a magnetostatic flux tube, in the Aharonov–Casher effect magnetic flux tubes wind around electric charges. [13] used magnetic dipoles (neutrons) to play the role of small magnetic flux tubes, while [14] used vortices in Josephson junction arrays. In this context which-path experiments can also be performed which partially (or totally) destroy the interference.

The aim of the present paper is to provide a unified approach to all these experiments. Their study is usually based on a Hilbert space of quasi-periodic functions $R(x+2\pi) = R(x)\exp(\mathrm{i}\theta)$ with fixed θ. The variable x is a coordinate here for the covering space of a circle. Clearly this Hilbert space is sufficient only when the phase θ has a fixed value; and it requires enlargement when the phase θ is uncertain, in order to accomodate the various values of θ. We define explicitly this enlarged Hilbert space and show how two dual bases which are related to each other through a Fourier transform, can be used for the description of Aharonov–Bohm and Aharonov–Casher phenomena, correspondingly. We also study quantitatively how in the case of which-path experiments, the interference is partially (or totally) destroyed.

In section 2 we consider the Aharonov–Bohm effect and enlarge its Hilbert space in order to be able to describe situations where the phase θ is not fixed (e.g. the decoherence in which-path experiments). Superselection rules which divide this enlarged Hilbert space into various magnetic sectors labelled by the magnetic flux at the centre, are studied. In section 3 we consider the Aharonov–Casher effect and explain its duality to the Aharonov–Bohm effect. Here we also show that an enlarged Hilbert space is needed in order to describe which-path experiments in this context. Dual superselection rules which divide this enlarged Hilbert space into various electric sectors labelled by the electric charge at the centre, are studied. In section 4 we present explicitly the enlarged Hilbert space and explain that the Aharonov–Bohm wavefunctions are related to the Aharonov–Casher wavefunctions through a Fourier transform. In section 5 we present the charge and flux operators which are dual to each other and act on this enlarged Hilbert space. In section 6 we study quantitatively the decoherence in which-path experiments. We conclude in section 7 with a discussion of our results and explain connections between our ideas and related ideas in other branches of physics.

2. The Aharonov–Bohm effect: electric charges winding around a magnetic flux tube

Here we briefly present the Aharonov–Bohm effect as applied to electron interference experiments in the presence of magnetostatic flux; then discuss the more general problem of electric charges moving on a circle threaded by a magnetostatic flux tube; and then enlarge the Hilbert space in order to be able to describe which-path experiments.

We consider the experiment shown in figure 1, where electric charges that follow two different paths c_0 and c_1 interfere, and the intensity is measured on the screen. A magnetostatic flux ϕ is threading the area between the two paths. The electrons feel a vector potential A_i, which is related to the magnetic flux ϕ through the relation:

$$\phi = \oint_c A_i\,\mathrm{d}x_i \tag{1}$$

where $c = c_1 - c_0$ is a counterclockwise contour around the origin.

Let R_0, R_1 be the wavefunctions corresponding to the two paths c_0 and c_1 (in the absence of magnetic flux). We neglect for simplicity here paths with higher winding numbers. The intensity at some point Δ on the screen is:

$$I(\Delta) = |R_0 + R_1\exp(\mathrm{i}e\phi)|^2 = |R_0|^2 + |R_1|^2 + 2|R_0||R_1|\cos(\sigma + e\phi) \tag{2}$$

where $\sigma = \arg(R_1) - \arg(R_0)$ (and in units where Planck's constant, the velocity of light in

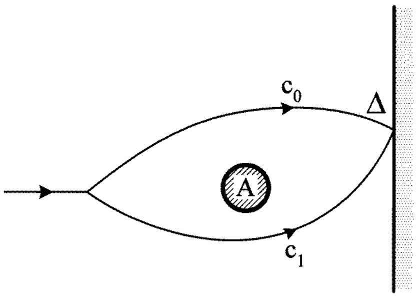

Figure 1. Electron interference in the presence of a magnetostatic flux in the region A, which is perpendicular to the plane of the diagram.

vacuum and Boltzmann's constant are equal to one: $\hbar = c = k_B = 1$). We consider the case of equal splitting in which $|R_0|^2 = |R_1|^2 = \frac{1}{2}$ and get:

$$I(\Delta) = 1 + \cos(\sigma + e\phi). \tag{3}$$

More generally we consider paths with all winding numbers. Let R_N be the wavefunction corresponding to the path c_N with winding number N. Then,

$$I(\Delta) = \left|\sum_N R_N \exp(\mathrm{i}eN\phi)\right|^2 = 1 + \sum_N a_N \cos(\sigma_N + eN\phi) \tag{4}$$

where a_N and σ_N can be expressed in terms of the R_N. It is seen that while in equation (3) $I(\Delta)$ is a sinusoidal function of ϕ with period $2\pi/e$, in equation (4) $I(\Delta)$ contains higher integral harmonics and therefore it is a general periodic function of ϕ with period $2\pi/e$. The visibility, in the case of equation (3) is 1, while in the case of equation (4) is in general less than 1 and it needs to be calculated numerically for each special case.

We next consider the wider problem of electric charges on a circle, with a magnetic flux tube threading the circle through its centre. The wavefunction $R(x)$ obeys the quasi-periodic boundary condition

$$R(x + 2\pi) = R(x)\exp(\mathrm{i}\theta) \tag{5}$$

where $\theta = e\phi$. It will be convenient from a mathematical point of view, to consider the case of magnetostatic flux which is a rational multiple of the flux quantum (fluxon):

$$\phi = \frac{2\pi}{e}\frac{s}{q} \tag{6}$$

where s, q are integers. In this case equation (1) becomes

$$R(x + 2\pi) = R(x)\exp\left(\mathrm{i}\frac{2\pi s}{q}\right). \tag{7}$$

We call $H(q; s)$ the Hilbert space of the functions obeying equation (7).

We now extend the above formalism, to include the which-path cases described in the introduction, where there is uncertainty in the phase in conjunction with partial (or even full) knowledge of the paths that the electrons follow. In equations (5), (7) the phase is fixed and this clearly is not suitable for physical situations where the phase is uncertain. For this reason we enlarge the Hilbert space $H(q; s)$ into the Hilbert space $H(q)$ which is the direct sum

$$H(q) = \sum_{s=0}^{q-1} H(q; s). \tag{8}$$

This space contains functions which obey equation (5) not with a fixed value of the phase, but with 'all' values. For mathematical convenience we consider only discrete values of the phase (with step $2\pi/q$), but in the limit $q \to \infty$ the phase takes all values. It is easily seen that $H(q)$ is the space of periodic functions with period $2\pi q$:

$$R(x + 2\pi q) = R(x). \tag{9}$$

From a physical point of view the periodicity $2\pi q$ is easily understood, because as the charges go from x to $x + 2\pi q$ they acquire extra phase which is an integer multiple of 2π. This is true even in the case that the phase is uncertain (provided that it takes only the discrete values $2\pi s/q$).

In [15] we have used this enlarged Hilbert space for a different problem, namely the study of the back-reaction from the charges on the circle, to the external magnetostatic flux. In this paper we use the Hilbert space $H(q)$ to describe physical situations where we have some knowledge about the paths of the electrons, some uncertainty in the phase and partial destruction of the interference.

Note that each $H(q; s)$ defines a different magnetic sector within the Hilbert space $H(q)$, in the sense that the electron wavefunctions in $H(q; s)$ determine through their global properties the magnetic flux $2\pi s/(eq)$ (which in this context is a topological quantity).

We next consider two states $|\psi(s_1)\rangle$ and $|\psi(s_2)\rangle$ in two different magnetic sectors $H(q; s_1)$ and $H(q; s_2)$, correspondingly ($s_1 \neq s_2$). Superselection rules [16, 17] state that for all local observables Θ

$$\langle\psi(s_1)|\Theta|\psi(s_2)\rangle = 0. \tag{10}$$

Indeed, consider the superposition of states

$$|R\rangle = \sum_{s=0}^{q-1} a(s)|\psi(s)\rangle. \tag{11}$$

A local observable cannot detect a rotation by 2π performed by the operator $U(2\pi)$:

$$\langle R|\Theta U(2\pi)|R\rangle = \langle R|\Theta|R\rangle. \tag{12}$$

Using the relation

$$U(2\pi)|R\rangle = \sum_{s=0}^{q-1} a(s) \exp\left(\mathrm{i}\frac{2\pi s}{q}\right)|\psi(s)\rangle \tag{13}$$

we prove equation (10). Another equivalent way of expressing the superselection rules is by saying that local measurements on the state $|R\rangle$ can measure the $|a(s)|^2$ but they cannot measure the relative phases $\mathrm{Arg}[a(s_1)] - \mathrm{Arg}[a(s_2)]$. Therefore in the context of the present experiment, superpositions of electron wavefunctions from different magnetic sectors in the Hilbert space, like the state $|R\rangle$, are not *physically* realizable; only *mixtures* of the type

$$\rho_1 = \sum_{s=0}^{q-1} |a(s)|^2 |\psi(s)\rangle\langle\psi(s)| \tag{14}$$

are physically realizable, and they describe which-path Aharonov–Bohm experiments. We will study such experiments in section 6. We refer to the above as magnetic superselection rules.

3. The Aharonov–Casher effect: magnetic flux tubes winding around an electric charge line

We present briefly here the Aharonov–Casher effect as applied to vortex interference experiments in the presence of electric charge; then discuss the more general problem of

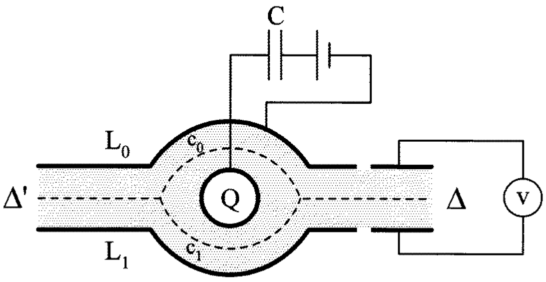

Figure 2. A two-dimensional array of superconducting islands coupled through Josephson junctions is enclosed by the boundaries L_0 and L_1. Vortices perpendicular to the plane of the diagram, travel ballistically from point Δ' to point Δ. The centre contains electric charge Q induced through the capacitor C, which is connected to a source. The voltage between the boundaries L_0 and L_1 (measured with the voltmeter), is proportional to the intensity of the magnetic vortices.

vortices moving on a cylinder with a charge line through its centre; and then enlarge the Hilbert space in order to describe which-path experiments. There is an electric–magnetic duality between the Aharonov–Casher case and the Aharonov–Bohm case and the presentation is such that this duality becomes clear.

The magnetic flux tubes are treated here as quantum mesoscopic objects and are assumed to be described by a wavefunction $S(x)$. Following [14] we consider the experiment of figure 2 where a two-dimensional array of superconducting islands coupled through Josephson junctions is enclosed by the boundaries L_0 and L_1. The Josephson coupling energy E_J is smaller than the charging energy E_C, so that the system is in the insulating phase and contains a condensate of vortices. Magnetic vortices perpendicular to the plane of the diagram, travel ballistically from point Δ' to point Δ. The centre contains electric charge per unit length Q induced through the capacitor C, which is connected to a source. The charge Q can be treated as a classical continuous variable. The magnetic vortices feel the 'dual potential' a_i which is related to the electric charge per unit length Q through the relation:

$$Q = \oint_c a_i \, \mathrm{d}x_i \tag{15}$$

where $c = c_1 - c_0$ is a counterclockwise contour around the origin. Note that like the covariant momentum of the electron is $p_i - eA_i$, the covariant momentum of the magnetic vortex per unit length is $p_i - \Phi_0 a_i$, where $\Phi_0 = \pi/e$ is the magnetic flux of the vortices (we take into account that Cooper pairs have charge $2e$).

Let $S_0(x)$ and $S_1(x)$ be the wavefunctions corresponding to vortices that follow the paths c_0 and c_1, correspondingly (in the absence of the electric charge Q). Below we use arguments very similar to those used for electron interference in the previous section, and write the total wavefunction and intensity (of vortices) at the point Δ as:

$$S(\Delta) = S_0 + \exp[\mathrm{i}\Phi_0 Q] S_1 \tag{16}$$

$$I(\Delta) = |S_0 + S_1 \exp(\mathrm{i}\Phi_0 Q)|^2 = |S_0|^2 + |S_1|^2 + 2|S_0||S_1| \cos(\tau + \Phi_0 Q) \tag{17}$$

where $\tau = \arg(S_1) - \arg(S_0)$. For equal splitting ($|S_0|^2 = |S_1|^2 = \frac{1}{2}$) this reduces to:

$$I(\Delta) = 1 + \cos(\tau + \Phi_0 Q). \tag{18}$$

Moving magnetic flux tubes create a voltage V between the boundaries L_0 and L_1 in figure 2 which is proportional to the intensity $I(\Delta)$. Therefore,

$$V = V_0[1 + \cos(\tau + \Phi_0 Q)]. \tag{19}$$

Equations (18) and (19) for vortices are analogous to equation (3) for electrons. We can also derive the analogue of equation (4), for paths with all winding numbers:

$$V = V_0 \left| \sum_N S_N \exp(\mathrm{i} N \Phi_0 Q) \right|^2 = V_0 \left[1 + \sum_N b_N \cos(\tau_N + N \Phi_0 Q) \right]. \qquad (20)$$

It is seen that while in equation (19) V is a sinusoidal function of Q with period $2e$, in equation (20) V contains higher integral harmonics and therefore it is a general periodic function of Q with period $2e$.

All the above results apply also to the Aharonov–Casher experiment with neutrons [13] in which case the covariant momentum is $p_i + a_i$ where $a_i = \epsilon_{ijk} \mu_j E_k$ is the dual potential, μ_j is the magnetic moment and E_k is the electric field generated by the line charge.

We next consider the more general problem of magnetic flux tubes moving on a cylinder (they always remain parallel to the axis of the cylinder). An electric charge line with uniform charge density, lies at the centre of the cylinder. The problem is effectively two-dimensional and the vortex wavefunction $S(x)$ obeys the quasi-periodic boundary condition

$$S(x + 2\pi) = S(x) \exp(\mathrm{i}\theta) \qquad (21)$$

where $\theta = \Phi_0 Q$. As we explained above, the charge is treated as a classical continuous variable, and for mathematical convinience is taken to be:

$$Q = 2e \frac{w}{q} \qquad (22)$$

where w, q are integers. In this case equation (21) becomes

$$S(x + 2\pi) = S(x) \exp\left(\mathrm{i} \frac{2\pi w}{q} \right). \qquad (23)$$

We call $H(q; w)$ the Hilbert space of the functions obeying equation (23). It is seen that equation (23) is very similar to equation (7). In both equations the exponential contains charge times magnetic flux. Note, however, that in equation (7) the magnetic flux takes fractional values of the flux quantum $2\pi/e$ and the electric charge is e. Here the electric charge per unit length takes fractional values of $2e$ (we take $2e$ for Cooper pairs) and the magnetic flux is π/e. This implies that the vortex wavefunctions $S(x)$ in the space $H(q; w)$, are related to the electron wavefunctions $R(x)$ in the space $H(q; s)$, through a finite Fourier transform. This Fourier transform provides a quantitative description of the duality between the electron (Aharonov–Bohm) wavefunctions and the vortex (Aharonov–Casher) wavefunctions; and it will be discussed in detail in the next section. It is already clear that the space $H(q; w)$ is different from the space $H(q; s)$.

We now extend the above formalism, to include the which-path cases where there is uncertainty in the phase in conjunction with partial (or even full) knowledge of the paths that the vortices follow. In equations (21), (23) the phase is fixed and this clearly is not suitable for physical situations where the phase is uncertain. For this reason we enlarge the Hilbert space $H(q; w)$ into the Hilbert space $H(q)$ which is the direct sum

$$H(q) = \sum_{w=0}^{q-1} H(q; w). \qquad (24)$$

This space contains functions which obey equation (21) not with a fixed value of the phase, but with ‘all’ values of the phase (for mathematical convenience we consider only discrete values of the phase). It is not difficult to see that the direct sum in equation (24) is the same as the direct sum in equation (8), and for this reason we used the same notation $H(q)$.

The $H(q; w)$ define various electric sectors within the Hilbert space $H(q)$, in the sense that the vortex wavefunctions in $H(q; w)$ determine through their global properties the electric charge at the centre $2ew/q$ (which in this context is a topological quantity).

We next consider two states $|\chi_1(w)\rangle$ and $|\chi_2(w)\rangle$ in two different electric sectors $H(q; w_1)$ and $H(q; w_2)$, correspondingly ($w_1 \neq w_2$). In this experiment superselection rules state that for all local observables Θ

$$\langle \chi(w_1)|\Theta|\chi(w_2)\rangle = 0. \tag{25}$$

Therefore, in the context of the present experiment, superpositions of vortex wavefunctions from two different electric sectors in the Hilbert space, like the state

$$|R'\rangle = \sum_{w=0}^{q-1} a(w)|\chi(w)\rangle \tag{26}$$

are not *physically* realizable; only *mixtures* of the type

$$\rho_2 = \sum_{w=0}^{q-1} |a(w)|^2 |\chi(w)\rangle\langle\chi(w)| \tag{27}$$

are physically realizable, and they describe which-path Aharonov–Casher experiments. We will study such experiments in section 6. We refer to the above as electric superselection rules.

We emphasize that the system considered in this section is in a different phase [18] from the system considered in the previous section. There we had one magnetic flux tube and electrons winding around it. Although that system was really in the 'Coulomb phase', the experiment simulated the 'superconducting phase' where Cooper pairs are winding around a vortex (which in this context is a topological object). In that case the wavefunction describes a condensate of Cooper pairs and its phase is related to the vector potential and the magnetic flux of the vortex. In this section the system is in the 'insulating phase' where we have vortices circulating around an electric charge (which in this context is a topological object). Here the wavefunction describes a condensate of vortices and its *dual* phase is related to the *dual* vector potential and the electric charge at the centre. Note that the superselection rules are different in these two phases. In the first case the magnetic superselection rule states that we can only have mixtures but not superpositions of electron states that belong to different magnetic sectors of the Hilbert space. In the second case the electric superselection rule states that we can only have mixtures but not superpositions of magnetic vortex states that belong to different electric sectors of the Hilbert space.

4. The Hilbert space for a unified approach

In the Hilbert space $H(q)$ we construct a formalism similar to the harmonic oscillator with dual variables the charge operator $\hat{w}$ and the magnetic flux operator $\hat{s}$. Note that in the experiment considered in section 2 the operator $\hat{w}$ can also be interpreted as winding number, i.e., the charge uncertainty is intimately related to the winding number uncertainty. Similarly, in the experiment considered in section 3 the operator $\hat{s}$ can also be interpreted as winding number, i.e., the the flux uncertainty is intimately related to the winding number uncertainty.

We consider a basis of wavefunctions $v(x, N, s)$ describing the case of a fixed flux s:

$$v(x; N, s) = \exp\left[\mathrm{i}\left(N + \frac{s}{q}\right)x\right]. \tag{28}$$

We will also denote them as $|N, s\rangle$. Note that $N + \frac{s}{q}$ are momenta consistent with the quasi-periodic boundary condition of equation (7). In the absence of magnetic flux the momenta take the integer values N, and the effect of the flux is to shift them from N to $N + \frac{s}{q}$. We also introduce the states

$$|x, s\rangle = \sum_{N=-\infty}^{\infty} v(x; N, s)|N, s\rangle \tag{29}$$

which can be interpreted in the context of the Aharonov–Bohm experiment discussed in section 2, as the usual position states for the charges on the circle, when the flux has a certain value s (i.e., zero uncertainty). Linear combinations of the states (29) (with fixed s and varying N)

$$|\psi(s)\rangle = \sum_{N=-\infty}^{\infty} a'(N; s)|N, s\rangle = \int \mathrm{d}x\, a'(x; s)|x, s\rangle \tag{30}$$

where

$$\sum_{N=-\infty}^{\infty} |a'(N; s)|^2 = \int \mathrm{d}x\, |a'(x; s)|^2 = 1 \tag{31}$$

give general wavefunctions characterized by fixed s (which belong in $H(q; s)$). In these states the flux is certain and the charge is uncertain. Physically they describe Aharonov–Bohm phenomena (in the absence of decoherence), which as we explained above are related to the superconducting phase.

We now consider a dual basis of wavefunctions $u(x, N, w)$ related to the $v(x, N, s)$ through the finite Fourier transform:

$$u(x, N, w) = q^{-1/2} \sum_{s=0}^{q-1} v(x, N, s) \exp\left(\mathrm{i}\frac{2\pi s w}{q}\right). \tag{32}$$

Here we form the superposition of all $v(x, N, s)$ with phase-factors $\exp(-\mathrm{i}e\phi w) = \exp(-\mathrm{i}\frac{2\pi s w}{q})$. We have shown in [15] that:

$$u(x; N, w) = q^{-1/2} \exp(\mathrm{i}Nx) \exp\left[\mathrm{i}\pi x \left(1 - \frac{1}{q}\right)\right] U_{q-1}\left[\cos\left(\frac{x + 2\pi w}{2q}\right)\right] \tag{33}$$

where U_{q-1} are Chebyshev polynomials of the second kind. We will also denote them as $|N, w\rangle$. We also introduce the states

$$|x, w\rangle = \sum_{s=0}^{q-1} \exp\left(\mathrm{i}\frac{2\pi s w}{q}\right)|x, s\rangle = \sum_{N=-\infty}^{\infty} u(x, N, w)|N, w\rangle \tag{34}$$

which can be interpreted as position states for the vortices in the Aharonov–Casher experiment discussed in section 3. Linear combinations of these states (with fixed w and varying N)

$$|\chi(w)\rangle = \sum_{N=-\infty}^{\infty} b'(N, w)|N, w\rangle = \int \mathrm{d}x\, b'(x; w)|x, w\rangle \tag{35}$$

where

$$\sum_{N=-\infty}^{\infty} |b'(N; w)|^2 = \int \mathrm{d}x\, |b'(x; w)|^2 = 1 \tag{36}$$

give general wavefunctions characterized by fixed w. In these states the charge is certain and the flux is uncertain. Physically they describe Aharonov–Casher phenomena (in the absence of decoherence), which as we explained above are related to the insulating phase.

It is seen that in the superconducting phase (or more generally in experiments which simulate the superconducting phase), the system is in one of the states $|\psi(s)\rangle$ or in mixtures of these states. In the insulating phase (or more generally in experiments which simulate the insulating phase), the system is in one of the states $|\chi(w)\rangle$ or in mixtures of these states.

5. Flux and charge operators

The flux operator is defined as

$$\hat{s} = \sum_{N,s} s|N, s\rangle\langle N, s| \tag{37}$$

and has eigenvalues s and eigenfunctions the $|\psi(s)\rangle$ of equation (30). The charge operator is defined as

$$\hat{w} = \sum_{N,w} w|N, w\rangle\langle N, w| \tag{38}$$

and has eigenvalues w and eigenfunctions the $|\chi(w)\rangle$ of equation (35).

Although clearly the formalism is similar to the Harmonic oscillator, we stress that here the $w - s$ phase space is a discretized torus. In [15] we have studied in detail the Heisenberg–Weyl group of discrete displacements:

$$\begin{aligned} E &= \exp\left(-\mathrm{i}\frac{2\pi}{q}\hat{s}\right) \qquad F = \exp\left(\mathrm{i}\frac{2\pi}{q}\hat{w}\right) \\ FE &= EF\exp\left(\mathrm{i}\frac{2\pi}{q}\right) \qquad E^q = F^q = \mathbf{1}. \end{aligned} \tag{39}$$

Since this group is discrete, the $\hat{w}$, $\hat{s}$ do not obey the commutation relation $\frac{2\pi}{q}[w, s] = i$. However, in the large q limit the discretized torus effectively becomes a continuum which locally is a plane. In this limit the $\hat{w}$, $\hat{s}$ can be assumed to obey the commutation relation $\frac{2\pi}{q}[w, s] = i$. We work in the large q limit and we use the commutation relation:

$$[W, S] = i \qquad W = \left(\frac{2\pi}{q}\right)^{1/2} w \qquad S = \left(\frac{2\pi}{q}\right)^{1/2} s. \tag{40}$$

W, S are rescaled charge and flux, which for finite q are discrete variables; but in the large q limit considered here, they become continuous variables.

We consider a density matrix ρ and define the phase uncertainty as

$$\Delta S = \{\mathrm{Tr}[\rho S^2] - [\mathrm{Tr}(\rho S)]^2\}^{1/2} \tag{41}$$

and the charge uncertainty as:

$$\Delta W = \{\mathrm{Tr}[\rho W^2] - [\mathrm{Tr}(\rho W)]^2\}^{1/2}. \tag{42}$$

In the large q limit considered here, we can write the uncertainty relation $\Delta S \Delta W \geqslant \frac{1}{2}$ (for smaller values of q the entropic uncertainty relations can be used).

6. Decoherence in which-path experiments

We consider an Aharonov–Bohm which-path experiment in which, due to external disturbances by the observer, the electric charges are in the mixed state of equation (14). This is realizable in the apparatus of figure 1, which, however, now contains a path observation mechanism. The intensity at some point Δ on the screen is:

$$I(\Delta) = \sum_{s=0}^{q-1}\langle x, s|\rho_1|x, s\rangle = \sum_{s=0}^{q-1}|a(s)|^2|\langle x, s|\psi(s)\rangle|^2 = \sum_{s=0}^{q-1}|a(s)|^2 I_s(\Delta) \tag{43}$$

where $I_s(\Delta) = |\langle x, s|\psi(s)\rangle|^2$ is the result given in equation (4) with $e\phi = 2\pi s/q$. This equation shows clearly how the visibility is destroyed in which-path experiments. For $|a(s)|^2 = 1/q$ and the simple case of paths with winding numbers 0 and 1 only (for which as we have seen in section 2 we get the maximum visibility), we use as $I_s(\Delta)$ the result of equation (3) and get

$$I(\Delta) = 1 + \frac{1}{q}\sum_{s=0}^{q-1}\cos\left(\sigma_s + \frac{2\pi s}{q}\right). \tag{44}$$

Assuming a constant σ_s we find that the sum is zero, $I(\Delta) = 1$ and the visibility is zero.

Very similar arguments can be presented for the Aharonov–Casher which-path experiment. This is realizable in the apparatus of figure 2, equipped with a path observation mechanism. Then

$$\begin{aligned} V &= V_0\left[\sum_{w=0}^{q-1}\langle x, w|\rho_2|x, w\rangle\right] = V_0\left[\sum_{w=0}^{q-1}|a(w)|^2|\langle x, w|\chi(w)\rangle|^2\right] \\ &= \sum_{w=0}^{q-1}|a(w)|^2 V_w \end{aligned} \tag{45}$$

where $V_w = V_0|\langle x, w|\chi(w)\rangle|^2$ is the result given in equation (20) with $Q = 2ew/q$. This equation shows clearly how the visibility is destroyed in which-path experiments. For $|a(w)|^2 = 1/q$ and the simple case of paths with winding numbers 0 and 1 only, we use as V_w the result of equation (19) and get

$$V = V_0\left[1 + \frac{1}{q}\sum_{w=0}^{q-1}\cos\left(\tau_w + N\Phi_0\frac{2ew}{q}\right)\right]. \tag{46}$$

Assuming a constant τ_w we find that the sum is zero, $V = V_0$ and the visibility is zero.

It should be emphasized that there is a difference between Aharonov–Bohm which-path experiments and Aharonov–Casher which-path experiments. In the former case the relative phase between the two electron wavefunctions is disturbed by the measuring apparatus and we get decoherence; in the latter case the relative *dual* phase between the two vortex wavefunctions is disturbed by the measuring apparatus and we get 'dual decoherence' (i.e. decoherence of the dual phase).

7. Discussion

The Aharonov–Bohm effect studies the interference of electric charges winding around a magnetic flux tube. It uses the electron wavefunction whose phase is related to the vector potential and the magnetic flux of the vortex. The Aharonov–Casher effect studies the interference of magnetic vortices winding around an electric charge line. It uses the vortex wavefunction whose dual phase is related to the dual vector potential and the electic charge at the centre. In both cases we can have which-path experiments where the measuring process destroys the relative phase and relative dual phase and leads to decoherence and dual decoherence, correspondingly.

We have considered a unified formalism for all these experiments. Our Hilbert space contains Aharonov–Bohm electron wavefunctions with various phases and is able to describe which-path experiments with decoherence. Equations (3), (4) give the results for the case where there is no decoherence; and equations (43), (44) for the case where we observe the paths and get decoherence.

The Hilbert space also contains Aharonov–Casher vortex wavefunctions with various phases and is able to describe which-path experiments, in this context also. Equations (19), (20) give the results for the case where there is no decoherence; and equations (45), (46) for the case where we observe the paths and get decoherence.

The Aharonov–Bohm electron wavefunctions are related to the Aharonov–Casher vortex wavefunctions through a Fourier transform. Which wavefunction is relevant for a certain experiment, depends on whether the experiment is in the superconducting or insulating phase. For experiments in the superconducting phase the Aharonov–Bohm electron wavefunctions are appropriate. In this case the magnetic superselection rule applies and which-path Aharonov–Bohm experiments are described by the density matrix ρ_1 of equation (14). For experiments in the insulating phase the Aharonov–Casher vortex wavefunctions are appropriate. In this case the electric superselection rule applies and which-path Aharonov–Casher experiments are described by the density matrix ρ_2 of equation (27).

Electric–magnetic duality similar to the one used in this paper, plays an important role in various branches of physics (Kramers–Wannier duality in condensed matter [19], t'Hooft duality in Gauge theories [20], duality in quantum hair of black holes [21], superconductor-insulator duality [18], etc). Therefore, it can be argued that the experiments discussed here, test ideas which are very fundamental for the whole of physics.

References

[1] Aharonov Y and Bohm D 1959 *Phys. Rev.* **115** 485
Chambers R G 1960 *Phys. Rev. Lett.* **5** 3
Olariu S and Popescu I I 1985 *Rev. Mod. Phys.* **57** 339
Peshkin M and Tonomura A 1989 *The Aharonov–Bohm Effect (Lecture Notes in Physics vol 340)* (Berlin: Springer)
[2] Laidlaw M G G and Morette-De Witt C 1971 *Phys. Rev.* D **3** 1375
Schulman L S 1971 *J. Math. Phys.* **12** 304
Schulman L S 1981 *Techniques and Applications of Path Integration* (New York: Wiley)
Dowker J S 1972 *J. Phys. A: Math. Gen.* **5** 936
[3] Acerbi F, Morchio G and Strocchi F 1993 *Lett. Math. Phys.* **27** 1
Acerbi F, Morchio G and Strocchi F 1993 *J. Math. Phys.* **34** 899
Narnhofer H and Thirring W 1993 *Lett. Math, Phys.* **27** 133
[4] Allman B E, Cimmino A, Klein A G, Opat G I, Kaiser H and Werner S A 1992 *Phys. Rev. Lett.* **68** 2409
Allman B E, Cimmino A, Klein A G, Opat G I, Kaiser H and Werner S A 1993 *Phys. Rev.* A **48** 1799
[5] Washburn S and Webb R A 1986 *Adv. Phys.* **35** 375
Aronov A G and Sharvin Y V 1987 *Rev. Mod. Phys.* **59** 755
[6] Feynman R, Leighton R and Sands M 1965 *The Feynman Lectures on Physics* vol 3 (New York: Addison-Wesley)
[7] Furry W H and Ramsey N F 1960 *Phys. Rev.* **110** 629
Ramsey N F 1993 *Phys. Rev.* A **48** 80
[8] Stern A, Aharonov Y and Imry Y 1990 *Phys. Rev.* A **40** 3436
[9] Scully M O, Englert B G and Walther H 1991 *Nature* **351** 111
Englert B G, Walther H and Scully M O 1992 *Appl. Phys.* B **54** 366
Storey P, Tan S, Collett M and Walls D 1994 *Nature* **367** 626
[10] Buks E, Schuster R, Heiblum M, Mahalu D and Umansky V 1998 *Nature* **391** 871
[11] Gurvitz S A 1997 *Phys. Rev.* B **56** 15 215
Aleiner I L, Wingreen N S and Meir Y 1997 *Phys. Rev. Lett.* **79** 3740
Levinson Y 1997 *Europhys. Lett.* **39** 299
[12] Aharonov Y and Casher A 1984 *Phys. Rev. Lett.* **53** 319
Reznik B and Aharonov Y 1989 *Phys. Rev.* D **40** 4178
Goldhaber A S 1989 *Phys. Rev. Lett.* **62** 482
[13] Cimmino A, Opat G I, Klein A G, Kaiser H, Werner S A, Arif M and Clothier R 1989 *Phys. Rev. Lett.* **63** 380
[14] van Wees B J 1990 *Phys. Rev. Lett.* **65** 255
Orlando T P and Delin K A 1991 *Phys. Rev.* B **43** 8717
Elion W J, Wachters J J, Sohn L L and Mooij J E 1993 *Phys. Rev. Lett.* **71** 2311
[15] Vourdas A 1997 *J. Phys. A: Math. Gen.* **30** 5195
[16] Streater R and Wightman A 1964 *PCT, Spin Statistics and All That* (New York: Benjamin)
[17] Bogoliubov N N, Logunov A A and Todorov R T 1975 *Introduction to Axiomatic Field Theory* (Reading, MA: Benjamin)

[18] Girvin S M 1996 *Science* **274** 524
van der Zant H S J, Elion W J, Geerligs L J and Mooij J E 1996 *Phys. Rev.* B **54** 10 081
Sondhi S L, Girvin S M, Canini J P and Shahar D 1997 *Rev. Mod. Phys.* **69** 315

[19] Kramers H A and Wannier G 1941 *Phys. Rev.* **60** 252
Onsager L 1944 *Phys. Rev.* **65** 117
Wannier G 1945 *Rev. Mod. Phys.* **17** 50

[20] t' Hooft G 1979 *Nucl. Phys.* B **153** 141
t' Hooft G 1981 *Proc. 1980 Scottish Universities Summer School* ed K C Bowler and D G Sutherland (Edinburgh: Redwood Burn)

[21] Preskill J and Kraus L M 1990 *Nucl. Phys.* B **341** 50
Coleman S, Preskill J and Wilczek F 1991 *Phys. Rev. Lett.* **67** 1975